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VARIANCE ESTIMATION THROUGH THE MEAN SQUARE SUCCESSIVE DIFFERENCES AND SAMPLE VARIANCE USING APRIORI INFORMATION

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Summary

Some estimators for estimating variance are derived using the prior knowledge of coefficient of Kurtosis by mean square successive differences and sample variance and their properties are studied.

Key words: Mean square successive differences, coefficient of Kurtosis, minimum mean squared error estimator.

Introduction

The uses of the mean square successive differences in place of the sample variance as a measure of dispersion in samples from normal population when the sample sequence is in temporal order, have been discussed by various authors including Von Neumann *et al.* [11], Moor [6], Kamat [3] and Giesser [1].

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be n observations with mean μ and θ for x_i and y_i, respectively. Then the radial sample variance (s^2) with constant means and $(\delta^2/2)$, the radial mean square successive difference subject to gradual shifts in both the population means are defined as follows:

$$s^2 = (n-1)^{-1}\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{n}(y_i - y)^2\right] \qquad (1.1)$$

and

$$\frac{\delta^2}{2} = \frac{1}{2(n-1)}\left[\sum_{i=1}^{n-1}(x_i - x_{i+1})^2 + \sum_{i=1}^{n-1}(y_i - y_{i+1})^2\right] \qquad (1.2)$$

where $\overline{x}$ and $\overline{y}$ are the sample means of x and y, respectively based on n observations.

Considering the combination of $(n-1)^{-1} \sum_{i=1}^{n-1} (x_i - x_{i+1})^2$ and $(n-1)^{-1} \sum_{i=1}^{n} (y_i - \overline{y})^2$ Nandi [7] proposed the following estimator

$$\frac{\delta_M^2}{2} = \frac{1}{2(n-1)}\left[\sum_{i=1}^{n-1} (x_i - x_{i+1})^2 + \sum_{i=1}^{n} (y_i - \overline{y})^2\right] \tag{1.3}$$

for the variance σ^2. For the usefulness of the study of $\frac{\delta_M^2}{2}$ the reader is referred to Nandi [7]. Morands [5] and Kamat [4] have also discussed some similar combinations in connection with the study of circular probable error.

The unbiased estimators of σ^2 are

$$T_1 = \delta_M^2/4 \tag{1.4}$$

$$T_2 = \delta^2/4 \tag{1.5}$$

and

$$T_3 = s^2/2 \tag{1.6}$$

The variance of T_i, i = 1, 2, 3 are respectively given by

$$V(T_1) = \frac{\mu_2^2}{8\, n\, (n-1)^2} (A\, \beta_2 - C) \tag{1.7}$$

$$V(T_2) = \frac{\mu_2^2}{4\, (n-1)^2} [\, (2n-3)\, \beta_2 + 1)\,] \tag{1.8}$$

and

$$V(T_3) = \frac{\mu_2^2}{2\,n\,(n-1)}\,[\,(n-1)\,\beta_2 - (n-3)\,] \tag{1.9}$$

where $A = (4n^2 - 7n + 2)$, $C = (2n^2 - 9n + 6)$, $\beta_2 = \frac{\mu_4}{\mu_2^2}$ and μ_2 and μ_4 are second and-fourth central moments.

In this paper we have proposed three minimum mean squared error estimators for estimating σ^2 exploiting the known value of coefficient of kurtosis β^2 as apriori information. The merits of suggested estimators are also examined.

2. *The Estimators*

We define the following estimators of σ^2:

$$T_4 = \alpha_1\,\delta_M^2 \tag{2.1}$$

$$T_5 = \alpha_2\,\delta^2 \tag{2.2}$$

$$T_6 = \alpha_3\,s^2 \tag{2.3}$$

where α_1's i=1, 2, 3 are suitably chosen constants to be determined such that mean squared errors (MSE's) of T_j, j = 4, 5, 6 are minimum and δ_M^2, δ^2 and s^2 are same as defined in section 1.

Such technique for improving estimates of population parameters has been first introduced by Searls [8] and later used by Singh et al. [9]. Hirano [2] and Singh [10], among others.

The MSE's of Tj, j = 4, 5, 6 are respectively given by

$$MSE(T_4) = \frac{\mu_2^2}{n(n-1)^2}\,[2\alpha_1^2\{A(\beta_2 + 2n - 1) + 2\,(3n-2)\} - 8n(n-1)^2\,\alpha_1 + n(n-1)^2] \tag{2.4}$$

$$MSE(T_5) = \frac{\mu_2^2}{(n-1)^2}\,[4\alpha_2^2\{(2n-3)\,\beta_2 + 4(n-1)^2 + 1\} - 8n(n-1)^2\,\alpha_2 + (n-1)^2] \tag{2.5}$$

$$\mathrm{MSE}(T_6) = \frac{\mu_2^2}{n(n-1)}\left[2\alpha_3^2\{(n-1)\beta_2 + 2n^2 - 3n+3\} - 4n(n-1)\alpha_3 + n(n-1)\right] \tag{2.6}$$

which are respectively minimized for the following choices of α_i $(i = 1, 2, 3)$:

$$\alpha_1 = \frac{2n(n-1)^2}{[A(\beta_2 + 2n - 1) + 2(3n-2)]} \tag{2.7}$$

$$\alpha_2 = \frac{(n-1)^2}{[(2n-3)\beta_2 + 4(n-1)^2 + 1]} \tag{2.8}$$

$$\alpha_3 = \frac{n(n-1)}{[\beta_2(n-1) + 2n^2 - 3n + 3]} \tag{2.9}$$

These lead to the following minimum MSE estimators of σ^2 :

$$T_4^* = \frac{2n(n-1)^2\,\delta_M^2}{[A(\beta_2 + 2n - 1) + 2(3n-2)]} \tag{2.10}$$

$$T_5^* = \frac{(n-1)^2\,\delta^2}{[(2n-3)\beta_2 + 4(n-1)^2 + 1]} \tag{2.11}$$

and

$$T_6^* = \frac{n(n-1)\,s^2}{[\beta_2(n-1) + 2n^2 - 3n + 3]} \tag{2.12}$$

Substituting the values of α_1, α_2 and α_3 from (2.7), (2.8) and (2.9) in (2.4), (2.5) and (2.6), respectively we obtain the MSE's of T_4^*, T_5^* and T_6^* as

$$\mathrm{MSE}(T_4^*) = \frac{\mu_2^2(A\beta_2 - C)}{[A(\beta_2 + 2n - 1) + 2(3n-2)]} \tag{2.13}$$

$$\mathrm{MSE}(T_5^*) = \frac{\mu_2^2[(2n-3)\beta_2 + 1]}{[(2n-3)\beta_2 + 4(n-1)^2 + 1]} \tag{2.14}$$

and

$$MSE(T_6^*) = \frac{\mu_2^2[\ \beta_2\,(n-1) - n + 3\ ]}{[\ \beta_2\,(n-1) + 2n^2 - 3\,(n-1)\ ]} \qquad (2.15)$$

The relative efficiency of an estimator T with respect to t is defined by

$$RE(T, t) = \frac{MSE(t)}{MSE(T)}$$

Thus we have

$$RE(T_4^*, T_1) = 1 + \frac{(A\beta_2 - C)}{8n(n-1)^2} \qquad (2.16)$$

$$RE(T_5^*, T_2) = 1 + \frac{[\ (2n-3)\ \beta_2 + 1)}{4(n-1)^2} \qquad (2.17)$$

and

$$RE(T_6^*, T_3) = 1 + \frac{[\ (n-1)\ \beta_2 - n + 3\ ]}{2n\ (n-1)} \qquad (2.18)$$

It follows from the above expressions that the derived minimum MSE estimators T_4^*, T_5^* and T_6^* are uniformly better than the corresponding unbiased estimators T_1, T_2 and T_3. To see the performance of the minimum MSE estimators T_4^*, T_5^* and T_6^*, the relative efficiencies RE(T_4^*, T_1), RE(T_5^*, T_2) and RE(T_6^*, T_3) are computed for various values of n and β_2 and shown in Tables 1, 2 and 3 respectively.

Table 1 exhibits that the RE(T_4^*, T_1) increases as β_2 increases and it decreases as n increases. For large sample size n and small β_2 both the estimators T_4^* and T_1 are equally efficient. However, there is little gain in RE(T_4^*, T_1) when n is large and $\beta_2 > 3$. Tables 2 and 3 also show a similar pattern of RE (T_5^*, T_2) and RE(T_6^*, T_3), respectively.

In order to compare the performance of Nandi's [7] estimator $\left(\frac{\delta_M^2}{2}\right)$, with T_4^* we have

$$V\left(\frac{\delta_M^2}{2}\right) = \frac{\mu_2^2}{2n\,(n-1)^2}\ (A\,\beta_2 - C)$$

Table 1 . Percent relative efficiency of T_4^* with respect to T_1 for various values of n and β_2

n \ β_2	1	2	3	4	5	6	8	10
3	120.80	138.54	156.25	173.96	191.67	209.38	244.79	280.20
5	108.80	119.22	129.69	140.16	150.63	161.09	182.03	202.97
10	103.33	108.46	113.58	118.70	123.83	128.95	131.33	149.44
20	101.45	103.98	106.51	109.04	111.57	114.10	119.17	124.23
30	100.92	102.60	104.28	105.96	107.64	109.32	112.68	116.04
40	100.67	101.93	103.19	104.45	105.70	106.96	109.48	111.99
50	100.53	101.54	102.54	103.55	104.55	105.56	107.57	109.58
60	100.44	101.27	102.11	102.95	103.79	104.62	106.30	107.97
100	100.26	100.76	101.26	101.76	102.26	102.76	103.77	147.69

Table 2 . Percent relative efficiency of T_5^* with respect to T_2 for various values of n and β_2

n \ β_2	1	2	3	4	5	6	8	10
3	125.00	143.75	162.50	181.25	200.00	218.75	256.25	293.75
5	112.50	123.44	134.38	145.31	156.25	167.19	189.06	210.94
10	104.69	109.11	113.54	117.97	122.40	126.82	135.68	144.53
20	102.63	105.11	107.76	110.32	112.88	115.44	120.57	125.69
30	101.72	103.42	104.69	106.25	107.81	109.36	112.47	115.58
40	101.28	102.55	103.81	105.08	106.34	107.61	110.14	112.67
50	101.02	102.03	103.04	104.05	105.06	106.07	108.09	110.11
60	100.85	101.69	102.53	103.37	104.21	105.04	106.73	108.41
100	100.51	101.01	101.51	102.01	102.52	103.02	104.02	105.03

Table 3 . Percent relative efficiency of T_6^* with respect to T_3 for different values of n and β_2

n \ β_2	1	2	3	4	5	6	8	10
3	116.67	133.33	150.00	166.67	183.33	200.00	233.33	266.67
5	105.00	115.00	125.00	135.00	145.00	155.00	175.00	195.00
10	101.11	106.11	111.11	116.11	121.11	126.11	136.11	146.11
20	100.26	102.76	105.26	107.76	110.26	112.76	117.76	122.76
30	100.15	101.76	105.12	106.78	106.78	108.45	111.78	115.12
40	100.06	101.31	103.81	105.06	105.06	106.64	109.14	111.31
50	100.04	101.04	103.04	104.04	104.04	105.04	107.04	109.04
60	100.03	100.86	101.69	102.53	103.36	104.19	105.86	107.53
100	100.01	100.51	101.01	101.51	102.01	103.00	103.51	104.51

Table 4 : Percent relative efficiency of T_4^* with respect to Nandi (1968) estimator $\left(\frac{\delta_m^2}{2}\right)$ for various values of n and β_2

n \ β_2	1	2	3	4	5	6	8	10
3	483.33	554.17	625.00	695.83	766.67	837.50	979.17	1120.83
5	435.00	476.88	518.75	576.88	602.50	644.38	728.13	811.88
10	425.43	433.83	454.32	427.82	507.41	515.80	556.79	609.88
20	405.79	415.91	426.04	442.99	453.12	456.41	476.67	503.74
30	403.68	410.40	417.12	423.85	430.57	437.29	450.73	464.18
40	402.69	407.72	412.76	416.34	422.82	427.75	437.91	447.99
50	402.08	406.10	410.12	414.14	218.16	422.18	433.16	438.26
60	401.75	405.10	408.45	411.79	415.14	418.49	425.18	431.88
100	401.03	403.04	405.04	407.05	409.05	211.06	415.07	419.08

Thus the relative efficiency of T_4^* with respect to $\frac{\delta_M^2}{2}$ is given by

$$RE\left(T_4^*, \frac{\delta_M^2}{2}\right) = 2 + \frac{[\,A(\beta_2 + n) - (3n-2)(n-3)\,]}{2n(n-1)^2} \tag{2.19}$$

which clearly demonstrates that the minimum MSE estimator T_4^* is more efficient than Nandi's estimators $\left(\frac{\delta_M^2}{2}\right)$. This fact can also be seen from Table 4. It follows from Table 4 that $RE\left(T_4^*, \frac{\delta_M^2}{2}\right)$ increases as β_2 increases for all n and it decreases as n increases for all β_2. For small samples size n and large values of β_2, larger gains are achieved.

3. *Robustness of the Estimators* T_4^*, T_5^* and T_6^*

In this section we study the 'robustness' of the estimators T_j^*, j = 4, 5, 6 against departure from the true value of coefficient of kurtosis β_2.

Sometimes, it is possible that there may not be exact information about β_2, and only a good guess of the value of β_2 is available i.e. $\tilde{\beta}_2 = \alpha\,\beta_2$ where β_2 is the true value and α is any positive constant indicating departure in true value.

Using $\tilde{\beta}_2$ in place of β_2 in T_4^*, T_5^* and T_6^* defined in (2.10), (2.11) and (2.12) respectively, we obtain the feasible estimators of σ^2 as

$$\hat{T}_4^{\bullet} = \frac{2n(n-1)^2 \ \delta_M^2}{[A(\beta_2 + 2n - 1) + 2(3n-2)]} \tag{3.1}$$

$$\hat{T}_5^{\bullet} = \frac{(n-1)^2 \ \delta^2}{[(2n-3)\beta_2 + 4(n-1)^2 + 1]} \tag{3.2}$$

and

$$\hat{T}_6^{\bullet} = \frac{n(n-1) \ s^2}{[\beta_2(n-1) + 2n^2 - 3n + 3]} \tag{3.3}$$

In order to have MSE $(\hat{T}_4^{\bullet}) \leq V(T_1)$, MSE $(\hat{T}_5^{\bullet}) \leq V(T_2)$ and MSE $(\hat{T}_6^{\bullet}) \leq V(T_3)$, we should have the following inequality :

$$\text{either} \quad \frac{[A\beta_2\{8n(n-1)^2 + B\} - BC]}{A\beta_2[8n(n-1)^2 + C - A\beta_2]} < \alpha < \frac{C}{A\beta_2}$$

$$\text{or} \quad \frac{C}{A\beta_2} < \alpha < \frac{[A\beta_2\{8n(n-1)^2 + B\} - BC]}{A\beta_2[8n(n-1)^2 + C - A\beta_2]} \tag{3.4}$$

$$0 < \alpha < \frac{[(2n-3)\beta_2\{4(n-1)^2 + b\} + b]}{(2n-3)\beta_2[(2n-3)(2n-1) - (2n-3)\beta_2]} \tag{3.5}$$

and

$$\text{either} \quad \frac{[(n-1)\beta_2(4n^2-5n+3) - (n-3)(2n^2-3n+3)]}{(n-1)\beta_2(2n^2-n-3-(n-1)\beta_2)} < \alpha < \frac{(n-3)}{(n-1)\ \beta_2}$$

$$\text{or} \quad \frac{(n-3)}{(n-1)\ \beta_2} < \alpha < \frac{[(n-1)\beta_2(4n^2-5n+3) - (n-3)(2n^2-3n+3)]}{(n-1)\beta_2(2n^2-n-3-(n-1)\beta_2)} \tag{3.6}$$

where $A = (4n^2 - 7n + 2)$, $B = (8n^2 - 18n^2 + 17n - 6)$, $C = (2n^2 - 9n + 6)$ and $b = (4n^2 - 8n + 5)$

We have computed the ranges of $\alpha(\%)$ for different values of n and β_2 and compiled in tables 5,6 and 7. From these tables we observe that the proposed estimators $\hat{T}_4^{\bullet}, \hat{T}_5^{\bullet}$ and $\hat{T}_6^{\bullet}$ have also smaller mean square errors than the corresponding unbiased estimators T_1, T_2 and T_3 for some guessed value of α, So we can prefer these estimators in the situations when some guessed value of α

Table 5 : The ranges of α (in percentage) for different values of n and β_2 for which $\hat{T}_4^*$ is better than T_1

n \ β_2		1	2	3	4	5	6	8	10
3	L	0	0	0	0	0	0	0	0
	U	279.57	345.31	478.15	672.41	2481.18	>0	>0	>0
5	L	16.42	8.21	5.47	4.11	3.28	2.74	2.05	0
	U	177.69	210.71	245.91	291.17	354.54	451.13	980.45	>1.642
10	L	34.94	17.47	11.65	8.74	6.99	5.82	4.37	3.49
	U	169.55	197.78	216.12	233.26	251.25	270.93	318.94	385.28
20	L	42.76	21.41	14.27	10.71	8.56	7.14	5.35	4.28
	U	158.86	185.10	197.67	207.05	215.37	223.36	239.53	256.93
30	L	45.28	22.64	15.09	11.32	9.06	7.55	5.66	4.53
	U	155.73	181.49	192.50	199.92	206.00	211.46	221.75	231.96
40	L	46.49	23.24	15.50	11.62	9.30	7.75	5.81	4.65
	U	154.24	179.78	190.07	196.60	201.68	206.06	213.91	221.34
50	L	47.20	23.60	15.73	11.80	9.44	7.87	5.90	4.72
	U	153.36	178.78	188.66	194.68	199.19	202.97	209.50	215.46
60	L	47.68	23.84	15.89	11.92	9.54	7.94	5.96	4.77
	U	152.79	178.13	187.74	193.43	197.58	201.20	206.68	211.73
100	L	48.61	24.31	16.20	12.15	9.72	8.10	6.08	4.86
	U	151.65	176.85	185.93	191.00	194.46	197.12	201.28	204.67

L = Lower range
U = Upper range

(nearer to the true value) is known. Further, to appreciate the idea of robustness, we computed the relative efficiencies of $\hat{T}_4^*, \hat{T}_5^*$ and $\hat{T}_6^*$ with respect to corresponding unbiased estimators T_1, T_2 and T_3 for various values of n, β_2 and α and presented in tables 8, 9 and 10. From tables 1, 2, 3, 8, 9, and 10, when $\tilde{\beta}_2$ is used in place of β_2, the loss in efficiency can easily be observed. It is seen from these tables that the loss in efficiency is more as the departure is more from the true value of α.

Remark 3.1 : Instead of taking $\tilde{\beta}_2 = \alpha\,\beta_2$, one can also take $\tilde{\beta}_2 = \beta_2 \pm \alpha$ or $\tilde{\beta}_2 = c\beta_2 + d$ (α, c, d being suitably chosen scalers) for constructing the estimators like $\hat{T}_j^*$, j = 4, 5, 6.

Table 6 : The ranges of α (in percentage) for different values of n and β_2 for which $\hat{T}_5^*$ is better than T_2

n \ β_2		1	2	3	4	5	6	8	10
3	L	0	0	0	0	0	0	0	0
	U	322.23	398.15	581.48	1147.23	>0	>0	>0	>0
5	L	0	0	0	0	0	0	0	0
	U	246.95	272.75	314.52	375.22	467.36	621.67	1859.49	>0
10	L	0	0	0	0	0	0	0	0
	U	218.34	227.88	240.95	256.39	274.30	295.11	348.34	425.44
20	L	0	0	0	0	0	0	0	0
	U	208.26	212.46	217.87	223.84	230.27	297.14	252.23	269.61
30	L	0	0	0	0	0	0	0	0
	U	205.39	208.02	211.40	215.08	219.00	223.07	231.73	241.14
40	L	0	0	0	0	0	0	0	0
	U	203.91	205.90	208.38	211.16	213.83	216.71	222.75	229.17
50	L	0	0	0	0	0	0	0	0
	U	203.13	204.68	206.64	208.72	210.89	213.12	217.63	222.62
60	L	0	0	0	0	0	0	0	0
	U	202.58	203.88	205.49	207.20	208.97	210.79	214.55	218.47
100	L	0	0	0	0	0	0	0	0
	U	201.53	202.29	203.24	204.24	205.27	206.31	208.45	210.64

Table 7 : The ranges of α (in percentage) for different values of n and β_2 for which $\hat{T}_6^*$ is better than T_3

n \ β_2		1	2	3	4	5	6	8	10
3	L	0	0	0	0	0	0	0	0
	U	240.00	300.00	400.00	600.00	1200.00	>0	>0	>0
5	L	50.00	25.00	16.67	12.50	10.00	8.33	6.25	5.00
	U	155.26	201.47	238.89	281.73	337.27	415.74	756.25	3805.00
10	L	77.78	38.89	25.93	19.44	15.56	12.96	9.72	7.78
	U	122.72	169.07	192.59	211.50	229.64	248.55	292.33	350.05
20	L	89.47	44.74	29.83	22.37	17.90	14.91	11.18	8.95
	U	110.58	158.40	177.97	190.70	200.89	209.99	227.18	244.72
30	L	93.10	46.56	31.04	23.28	18.62	15.52	11.64	9.31
	U	106.91	155.42	173.89	185.00	193.22	200.08	211.96	222.99
40	L	94.87	47.44	31.62	23.72	18.97	15.81	11.86	9.49
	U	105.14	153.96	171.98	182.33	189.67	195.54	205.18	213.61
50	L	95.92	47.96	31.97	23.98	19.18	15.99	11.99	9.59
	U	104.09	153.14	170.86	180.79	187.62	192.93	201.34	208.38
60	L	96.61	48.31	32.20	24.15	19.32	16.10	12.08	9.66
	U	103.39	152.59	170.13	179.78	186.29	191.25	198.87	205.05
100	L	97.98	48.99	32.66	24.50	19.60	16.30	12.25	9.80
	U	102.02	151.53	168.71	177.82	183.70	187.98	194.14	198.72

Table 8 : Percent relative efficiency of $\hat{T}_4^*$ with respect to T_1 for various values of n, β_2 & α

α	n \ β_2	1	2	3	4	5	6	8	10
0.50	3	117.31	131.23	145.41	159.84	174.49	189.36	219.64	250.54
0.75		120.00	136.89	153.89	170.99	188.17	205.41	240.05	274.84
1.25		120.11	137.26	154.56	171.87	189.45	206.98	242.12	277.34
1.50		118.19	134.07	150.60	167.48	184.58	201.82	236.56	271.52
0.50	5	105.66	113.74	121.90	130.16	138.52	146.98	164.17	181.66
0.75		108.00	117.93	127.90	137.92	147.98	158.08	178.36	198.74
1.25		108.07	118.13	128.28	138.49	148.75	159.04	179.70	200.43
1.50		106.19	115.31	124.78	134.50	144.38	154.37	174.59	195.01
0.50	10	101.37	105.39	109.34	113.30	117.28	121.28	129.26	137.54
0.75		102.85	107.71	112.56	117.43	122.30	127.19	136.99	146.84
1.25		102.87	107.78	112.69	117.63	122.58	127.56	137.55	147.58
1.50		101.55	105.91	110.32	114.83	119.42	124.09	133.59	143.24
0.50	20	100.34	102.38	104.32	106.26	108.19	110.13	114.04	117.96
0.75		101.17	103.58	105.97	108.36	110.76	113.15	117.95	122.77
1.25		101.18	103.60	106.01	108.42	110.84	113.27	118.14	123.04
1.50		100.40	102.52	104.61	106.72	108.86	111.04	115.47	119.99
0.50	30	100.15	101.52	102.81	104.09	105.37	106.65	109.22	111.80
0.75		100.73	102.33	103.92	105.50	107.09	108.67	111.85	115.03
1.25		100.73	102.34	103.94	105.53	107.13	108.73	111.94	115.17
1.50		100.18	101.59	102.94	104.31	105.69	107.09	109.94	112.84
0.50	40	100.09	101.11	102.08	103.04	103.99	104.95	106.86	108.78
0.75		100.53	101.73	102.91	104.10	105.28	106.47	108.84	111.23
1.25		100.53	101.73	102.92	104.12	105.31	106.50	108.90	111.30
1.50		100.10	101.15	102.16	103.16	104.18	105.21	107.29	109.41
0.50	50	100.05	100.88	101.65	102.41	103.18	103.94	105.47	106.99
0.75		100.41	101.37	102.32	103.27	104.21	105.16	107.05	108.95
1.25		100.41	101.38	102.33	103.28	104.23	105.18	107.09	109.00
1.50		100.06	100.90	101.70	102.50	103.30	104.11	105.75	107.42
0.50	60	100.04	100.73	101.37	102.00	102.64	103.27	104.54	105.81
0.75		100.34	101.14	101.93	102.71	103.50	104.29	105.87	107.44
1.25		100.34	101.14	101.93	102.72	103.51	104.31	105.89	107.48
1.50		100.05	100.74	101.40	102.06	102.73	103.39	104.75	106.12
0.50	100	100.01	100.43	100.81	101.19	101.57	101.95	102.71	103.47
0.75		100.20	100.68	101.15	101.62	102.09	102.51	103.51	104.45
1.25		100.20	100.68	101.15	101.62	102.10	102.57	103.52	104.46
1.50		100.02	100.44	100.83	101.21	101.60	102.00	102.79	103.58

Table 9 : Percent relative efficiency of $\hat{T}_5^*$ with respect to T_2 for various values of n, β_2 & α.

α	n \ β_2	1	2	3	4	5	6	8	10
0.50	3	121.80	136.72	151.86	167.24	182.86	198.70	230.94	263.83
0.75		124.25	142.17	160.20	178.33	196.54	214.32	251.52	288.38
1.25		124.35	142.53	160.87	179.31	197.82	216.39	253.61	290.91
1.75		120.09	135.46	152.16	169.55	187.33	205.35	241.84	278.68
0.50	5	110.20	118.65	127.22	135.89	144.68	153.56	171.60	189.96
0.75		111.94	122.32	132.74	143.22	153.74	164.29	185.50	206.80
1.25		112.00	122.50	133.09	143.76	154.47	165.22	186.81	208.46
1.75		108.49	116.59	125.62	135.19	145.10	155.24	175.95	197.01
0.50	10	104.34	108.33	112.34	116.39	120.47	124.58	132.88	141.27
0.75		105.26	110.20	115.17	120.14	125.14	130.14	140.18	150.27
1.25		105.27	110.26	115.27	120.33	125.40	130.49	140.71	150.99
1.75		103.16	106.47	110.20	114.22	118.46	122.87	132.03	141.53
0.50	20	102.01	103.01	105.95	107.84	109.80	111.76	115.71	119.69
0.75		102.48	104.89	107.03	109.73	112.14	114.56	119.42	124.30
1.25		102.48	104.90	107.33	109.77	112.22	114.67	119.60	124.55
1.75		101.33	102.74	104.29	105.96	107.73	109.59	113.51	117.64
0.50	30	101.31	102.59	103.87	105.15	106.44	107.73	110.32	112.92
0.75		101.62	103.21	104.81	106.40	108.00	109.60	112.80	116.01
1.25		101.62	103.22	104.82	106.43	108.04	109.65	112.89	116.14
1.75		100.84	101.71	102.65	103.66	104.73	105.85	108.21	110.71
0.50	40	100.97	101.92	102.88	103.83	104.79	105.75	107.68	109.61
0.75		101.20	102.39	103.58	104.77	105.96	107.16	109.54	111.94
1.25		101.21	102.40	103.59	104.79	105.99	107.19	109.60	112.01
1.75		100.61	101.24	101.91	102.62	103.37	104.15	105.80	107.55
0.50	50	100.77	101.53	102.29	103.05	103.81	104.58	106.11	107.65
0.75		100.96	101.91	102.85	103.80	104.75	105.71	107.61	109.51
1.25		100.96	101.91	102.86	103.81	104.77	105.73	107.64	109.57
1.75		100.47	100.97	101.49	102.03	102.61	103.20	104.45	105.78
0.50	60	100.64	101.27	101.90	102.54	103.17	103.81	105.08	106.35
0.75		100.80	101.58	102.37	103.16	103.95	104.74	106.33	107.91
1.25		100.80	101.59	102.38	103.17	103.96	104.76	106.35	107.95
1.75		100.39	100.79	101.22	101.66	102.12	103.09	103.60	104.65
0.50	100	100.38	100.76	101.14	101.89	102.27	102.27	103.03	103.79
0.75		100.47	100.95	101.42	101.89	102.36	102.83	103.78	104.72
1.25		100.47	100.95	101.42	101.89	102.36	102.84	103.79	104.74
1.75		100.23	100.46	100.70	100.95	101.20	101.47	102.01	102.58

Table 10 : Percent relative efficiency of $\hat{T}_6^*$ with respect to T_3 for various values of n, β_2 & α

α	n \ β_2	1	2	3	4	5	6	8	10
0.50	3	112.67	125.64	138.89	152.38	166.09	180.00	208.33	237.25
0.98		116.66	133.32	149.99	166.65	183.32	199.98	233.31	266.64
1.02		116.66	133.32	149.99	166.65	183.31	199.98	233.31	260.64
1.50		113.64	128.57	144.12	160.25	176.09	192.31	225.17	257.90
0.50	5	100.50	108.44	116.35	124.26	132.25	140.31	156.66	173.31
0.98		104.99	114.99	124.99	134.99	144.99	154.98	174.98	194.98
1.02		104.99	114.99	124.99	134.99	144.99	154.98	174.98	194.98
1.05		100.83	110.29	119.51	128.87	138.35	147.93	167.28	186.81
0.50	10	95.58	102.03	106.11	110.05	113.96	117.88	125.78	133.75
0.98		101.10	106.11	111.10	116.10	121.10	126.10	136.10	146.10
1.02		101.10	106.11	111.10	116.10	121.10	126.10	136.10	146.10
1.50		96.08	102.71	107.25	111.75	116.29	120.88	130.18	139.65
0.50	20	94.54	100.50	102.60	104.57	106.52	108.45	112.32	116.20
0.98		100.25	102.76	105.26	107.76	110.26	112.76	117.76	122.76
1.02		100.25	102.76	105.26	107.76	110.26	112.76	117.76	122.75
1.05		94.80	100.71	102.95	105.10	107.26	109.43	113.84	118.32
0.50	30	94.33	100.22	101.64	102.96	104.25	105.53	108.09	110.66
0.98		100.10	101.78	103.45	105.11	106.78	108.44	111.78	115.11
1.02		100.10	101.78	103.45	105.11	106.78	108.44	111.78	115.11
1.50		94.51	100.35	101.80	103.21	104.61	106.01	108.86	111.75
0.50	40	94.25	100.08	100.94	101.73	102.54	103.27	104.80	106.33
0.98		100.07	101.31	102.56	103.81	105.06	106.31	108.81	111.31
1.02		100.07	101.31	102.56	103.81	105.06	106.31	108.81	111.31
1.05		94.40	100.18	101.29	102.33	103.36	104.40	106.48	108.60
0.50	50	94.25	100.08	100.94	101.73	102.50	103.27	104.80	106.33
0.98		100.02	101.04	102.04	103.04	104.04	105.04	107.04	109.04
1.02		100.62	101.04	102.04	103.04	104.04	105.04	107.04	109.04
1.50		94.33	100.12	101.00	101.83	102.64	103.46	105.10	106.77
0.50	60	94.18	100.06	100.77	101.43	102.08	102.72	103.99	105.26
0.98		100.05	100.86	101.65	102.53	103.36	104.19	105.86	107.53
1.02		100.05	100.86	101.69	102.53	103.36	104.19	105.86	107.53
1.50		94.31	100.81	100.82	101.50	102.85	103.53	104.81	105.58
0.50	100	94.20	100.02	100.45	100.85	101.24	101.62	102.38	103.14
0.98		100.10	100.51	101.01	101.51	102.01	102.51	103.51	104.51
1.02		99.90	100.51	101.01	101.51	102.01	102.51	103.51	104.51
1.50		94.20	100.03	100.47	100.87	101.27	101.67	102.46	103.26

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